

A Dissipative Model of Plasma Equilibrium in Toroidal Systems

H. Wobig

Max-Planck-Institut für Plasmaphysik, EURATOM Association, Garching

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In order to describe a steady-state plasma equilibrium in tokamaks, stellarators or other non-axisymmetric configurations, the model of ideal MHD with isotropic plasma pressure is widely used:

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B}.$$

The ideal MHD-model of a toroidal plasma equilibrium requires the existence of closed magnetic surfaces. Several numerical codes have been developed in the past to solve the three-dimensional equilibrium problem, but so far no existence theorem for a solution has been proved. Another difficulty is the formation of magnetic islands and field line ergodisation, which can only be described in terms of ideal MHD if the plasma pressure is constant in the ergodic region. In order to describe the formation of magnetic islands and ergodisation of surfaces properly, additional dissipative terms have to be incorporated to allow decoupling of the plasma and magnetic field. In a collisional plasma, viscosity and inelastic collisions introduce such dissipative processes. In the model used a friction term proportional to the velocity \mathbf{v} of the plasma is included. Such a term originates from charge exchange interaction of the plasma with a neutral background. With these modifications, the equilibrium problem reduces to a set of quasilinear elliptic equations for the pressure, the electric potential and the magnetic field. The paper deals with an existence theorem based on the Fixed-Point method of Schauder. It can be shown that a self-consistent and unique equilibrium exists if the friction term is large and the plasma pressure is sufficiently low. The essential role of the dissipative terms is to remove the singularities of the ideal MHD model on rational magnetic surfaces. The problem has a strong similarity to Bénard cell convection, and consequently a similar behaviour such as bifurcation and exchange of stability is expected.

1. Introduction

In ideal MHD theory the simplest model of a plasma equilibrium is the one-fluid model with a scalar pressure p . The pressure gradient is balanced by the electromagnetic force $\mathbf{j} \times \mathbf{B}$:

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B}; \quad \nabla \times \mathbf{B} = \mathbf{j}. \quad (1)$$

In configurations with one negligible coordinate (axisymmetry or helical symmetry) this nonlinear problem leads to a Grad-Shafranov type of equation, which is nonlinear and elliptic. In general toroidal equilibrium there is no such symmetry and so far no existence theorem for solutions of (1) has been established. Nonetheless codes have been developed which calculate approximate solutions of (1). The basic problem in 3 dimensions is the existence of toroidally closed and nested surfaces, which is required in order to find nested pressure surfaces.

This strong coupling of pressure surfaces and magnetic surfaces is the result of

$$\mathbf{B} \cdot \nabla p = 0. \quad (2)$$

The low- β expansion as proposed by Spitzer [1] tries to solve the system (1) by the following iterative process:

$$\begin{cases} 0 = -\nabla p_n + \mathbf{j}_{n+1} \times \mathbf{B}_n, & \nabla \cdot \mathbf{j}_{n+1} = 0 \\ \nabla \times \mathbf{B}_{n+1} = \mathbf{j}_{n+1}, & \nabla \cdot \mathbf{B}_{n+1} = 0 \end{cases} \quad (3)$$
$$n = 1, 2, \dots$$

In every iteration step p_n is obtained from Equation (2).

There is no proof that magnetic surfaces always exist during this process. Furthermore, on rational magnetic surfaces – the rotational transform is a rational number – the well-known condition $\oint d\ell/B = \text{const}$ has to be satisfied if $\nabla p \neq 0$ on this surface. Otherwise the current density \mathbf{j} on rational surfaces diverges. As discussed by Grad [2], this constraint leads to a pathological pressure distribution with $p'(\psi) = 0$ on all rational magnetic surfaces. The

Reprint requests to Herrn Dr. H. Wobig, Max-Planck-Institut für Plasmaphysik, Boltzmannstraße 2, D-8046 Garching b. München.

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derivative of p with respect to the flux coordinate ψ is a discontinuous function. In a collisional plasma, however, the pressure distribution is the result of a diffusion process. Diffusion processes are usually described by second-order differential equations with continuous and continuously differentiable solutions. The diffusion process eliminates the discontinuities of the ideal model and results in $p(x)$ with continuous first-order and second-order derivatives.

In order to describe a steady-state plasma these collisional processes should be included. In the ideal MHD model inertial terms and viscosity terms are neglected. Both terms lead to a small variation of the plasma pressure in the magnetic surfaces. Another dissipative process is momentum exchange via charge exchange between the plasma and a neutral background. For simplicity we assume the neutral background to be at rest, then the momentum loss term of the plasma is proportional to the macroscopic flow velocity \mathbf{v} . In a steady-state plasma this friction term is always present since the plasma losses have to be balanced by some kind of refuelling mechanism which introduces neutral gas into the plasma. Ionisation of this neutral gas is the source term S of the plasma density.

Thus, in general, Eq. (1 a) should be replaced by

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \mathbf{v} + \alpha \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B}, \quad \nabla \times \mathbf{B} = \mathbf{j} \quad (4)$$

with ρ = mass density, $\nabla \mathbf{v}$ = viscosity term, $\alpha \mathbf{v}$ = friction term. Here $\alpha = nm v_{i0}$, where v_{i0} is the charge exchange frequency depending on the neutral gas density and n is the plasma density. Both viscosity and charge exchange friction describe momentum losses of the plasma or, in mathematical terms, both operators are hermitian and positive definite. Both effects are small compared with ∇p and $\mathbf{j} \times \mathbf{B}$. The essential role of these terms is to modify the momentum balance parallel to the magnetic field. The two effects only differ in their mathematical structure. Inertial terms and viscosity terms contain first-order and second-order derivatives of the velocity \mathbf{v} . Including these terms would increase the mathematical complexity of the problem appreciably.

The simplest change of the ideal MHD picture is to take just frictional dissipation into account. Then (4) is algebraic in the velocity \mathbf{v} . The plasma losses have to be balanced by a source term $S > 0$, and the

complete set of equations of the dissipative model is

$$\nabla p = \mathbf{j} \times \mathbf{B} - \alpha \mathbf{v}, \quad (5)$$

$$\nabla \Phi = \mathbf{v} \times \mathbf{B} - \eta \mathbf{j}, \quad (6)$$

$$\nabla \cdot \mathbf{j} = 0, \quad \nabla \cdot n \mathbf{v} = S, \quad (7)$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0 \quad (8)$$

with Φ = electric potential, η = Spitzer resistivity.

The friction term decouples the plasma and magnetic field since

$$\mathbf{B} \cdot \nabla p = -\alpha \mathbf{v} \cdot \mathbf{B} \neq 0.$$

The pressure and density are correlated by $p = nkT$. In principle the temperature is determined by the equation of thermal conduction, which allows one to calculate T if \mathbf{v} , n and the heat source are given. For the sake of simplicity, however, we only consider the case of an isothermal plasma with $T = \text{const}$. Then we could replace kT by 1 and n by p or vice versa.

The procedure of solving the system (5)–(7) is the following:

Equations (5), (6), (7) are solved for \mathbf{j} , \mathbf{v} , p with \mathbf{B} given. Then from (8) we obtain a magnetic field \mathbf{B}_1 . This procedure provides us with a mapping F which maps a set of functions \mathbf{B} into another set of fields \mathbf{B}_1 :

$$F: \mathbf{B} \rightarrow \mathbf{B}_1.$$

The system (6)–(8) has a solution if this mapping has a fixed point. The solution of (8) with \mathbf{j} given is a standard problem of electrodynamics. With the boundary condition $\mathbf{B} \rightarrow 0$ at infinity, the solution of (8) is

$$\mathbf{B}_1 = \nabla \times \iint_G \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}'.$$

For any Hölder-continuous function $\mathbf{j}(\mathbf{x}')$ the solution \mathbf{B}_1 exists and has continuous derivatives which are also Hölder continuous with the same coefficient. The two more difficult steps of the problem are to prove a solution of (5)–(7) with \mathbf{B} given and to prove the existence of a fixed point of the mapping F . In solving (5)–(7), no constraints on the magnetic field have to be imposed except that the field has continuous derivatives of the first order. This is satisfied for any magnetic field produced by continuous currents. Neither is the magnetic field required to have magnetic surfaces nor does the

distinction between rational and irrational magnetic surfaces play any role. Certainly the solution of the system will depend critically on the islands formation or ergodicity of the field \mathbf{B} . The basic problem is to show under what conditions solutions of (5), (6), (7) exist. This problem has to be solved in every step of the low- β expansion. The following section is devoted to an existence theorem for solutions to (5), (6), (7) with \mathbf{B} given.

2. Basic System

Since (5) and (6) are algebraic in \mathbf{j} and \mathbf{v} , the system can easily be inverted with respect to \mathbf{v} and \mathbf{j} :

$$\left. \begin{aligned} \mathbf{j} &= -\frac{1}{B^2 + \alpha\eta} \alpha \nabla_{\perp} \Phi - \sigma \nabla_{\parallel} \Phi - \frac{1}{B^2 + \alpha\eta} \nabla p \times \mathbf{B} \\ \mathbf{v} &= -\frac{1}{B^2 + \alpha\eta} \eta \nabla_{\perp} p - \frac{1}{\alpha} \nabla_{\parallel} p - \frac{1}{B^2 + \alpha\eta} \nabla \Phi \times \mathbf{B} \end{aligned} \right\} \quad (9)$$

with

$$\nabla_{\parallel} = \frac{\mathbf{B}}{B^2} (\mathbf{B} \cdot \nabla \dots), \quad \nabla_{\perp} = \nabla - \nabla_{\parallel}.$$

Inversion of the system is only possible if both coefficients α and η remain finite. In the case of $\alpha \rightarrow 0$ or $\eta \rightarrow 0$ the pressure p or the potential Φ must be constant on magnetic surfaces, which again implies strong coupling between the plasma and magnetic field. The velocity \mathbf{v} can be interpreted as the sum of a perpendicular diffusion velocity with the diffusion coefficient

$$D_{\perp} = \frac{n\eta}{B^2 + \alpha\eta}$$

and a parallel diffusion velocity with coefficient $D_{\parallel} = n/\alpha$.

$$\mathbf{v}_c = \frac{\nabla \Phi \times \mathbf{B}}{B^2 + \alpha\eta} \quad (10)$$

is the convective velocity arising from the electric field. Inserting (9) into (7) yields two equations for p and Φ :

$$\left\{ \begin{aligned} -\nabla \cdot \left\{ \frac{\alpha}{B^2 + \alpha\eta} \nabla_{\perp} \Phi + \sigma \nabla_{\parallel} \Phi \right\} &= \nabla \cdot \frac{\nabla p \times \mathbf{B}}{B^2 + \alpha\eta} \\ -\nabla \cdot \left\{ \frac{\eta n}{B^2 + \alpha\eta} \nabla_{\perp} p + \frac{n}{\alpha} \nabla_{\parallel} p \right\} - \nabla \cdot n \mathbf{v}_c &= S \end{aligned} \right. \quad (11)$$

For further study it is convenient to introduce dimensionless variables. As a length scale we introduce the average minor radius a of the toroidal domain G . B_0 is a reference magnetic field and $\Phi =: \sigma \Phi / B_0$ is the dimensionless electric potential. The following substitutions are made:

$$\begin{aligned} \frac{\mathbf{x}}{a} &\rightarrow \mathbf{x}, \quad \frac{p}{B_0^2} \rightarrow p, \quad \frac{n}{B_0^2} \rightarrow n, \quad \frac{\sigma \Phi}{B_0^2} \rightarrow \Phi, \\ \frac{\mathbf{B}}{B_0} &\rightarrow \mathbf{B}, \quad \frac{S a^2}{\eta B_0^2} \rightarrow S, \quad \frac{\eta \alpha}{B_0^2} \rightarrow \varepsilon g(x) p. \end{aligned}$$

From the explicit form of the Spitzer resistivity η and the friction coefficient α we derive for $\varepsilon g p$

$$\varepsilon g p = \frac{v_{e,i} \cdot v_{i,o}}{\Omega_e \cdot \Omega_i},$$

where $v_{e,i}$ is the electron-ion collision frequency and Ω the gyrofrequency. The factor p arises from the density dependence of the collision frequency $v_{e,i}$ and $g(x)$ describes the spatial variation of the neutral background and the magnetic field. ε indicates the magnitude of the charge exchange process, in a strong magnetic field it is a very small number. As a measure of the neutral background the function $g(x)$ is positive and bounded.

With these definitions, only one scaling parameter ε and the modified source function S remain in the equations. In dimensionless terms the equations are written

$$\left\{ \begin{aligned} -\nabla \cdot \left\{ \frac{\varepsilon g(x) p}{B^2 + \varepsilon g(x) p} \nabla_{\perp} \Phi + \nabla_{\parallel} \Phi \right\} \\ = \nabla \cdot \frac{\nabla p \times \mathbf{B}}{B^2 + \varepsilon g(x) p} \\ -\nabla \cdot \left\{ \frac{p}{B^2 + \varepsilon g(x) p} \nabla_{\perp} p + \frac{1}{g(x) \varepsilon} \nabla_{\parallel} p \right\} \\ - \nabla \cdot p \mathbf{v}_c = S \end{aligned} \right. \quad (12)$$

The dimensionless convective velocity \mathbf{v}_c is defined by

$$\mathbf{v}_c = \frac{\nabla \Phi \times \mathbf{B}}{B^2 + \varepsilon g(x) p}.$$

3. Boundary Conditions

In order to solve the system (12) boundary conditions have to be imposed. The coupled system (12)

is quasilinear and of second order in the variables. We consider a domain G with the boundary Γ . The surface Γ could be the last magnetic surface of a vacuum magnetic field or the boundary of a vacuum tube. We assume Γ to be a smooth surface with continuous second derivatives. The specific boundary conditions for p and Φ are the result of a boundary sheath model and cannot be given without further specification of such a model. The simplest case would be a metallic wall as a boundary with a constant potential Φ_r . Without loss of generality we may assume a vanishing potential $\Phi_r = 0$ on the boundary. This represents the toroidal equilibrium of a stellarator without loop voltage. Since the normal derivatives of the potential on a conducting wall are not zero in general, a current \mathbf{j} flows between the plasma and wall. In the case of a nonconducting wall the appropriate boundary condition is a vanishing plasma current into the boundary Γ . From $\mathbf{j} \cdot \mathbf{N} = 0$ the resulting boundary conditions of the potential are

$$\mathbf{N} \cdot \left\{ \frac{1}{B^2 + \alpha \eta} \alpha \nabla_{\perp} \Phi + \sigma \nabla_{\parallel} \Phi + \frac{1}{B^2 + \alpha \eta} \nabla p \times \mathbf{B} \right\} = 0.$$

If the pressure p is constant on the boundary, the last term in the brackets vanishes. In the case of these Neumann type boundary conditions the solution for Φ can be multivalued in the toroidal coordinate, thus representing an equilibrium with a toroidal loop voltage. This loop voltage drives a toroidal net current as in a tokamak equilibrium.

With respect to p either Dirichlet or Neumann-type boundary conditions could be required. In the case of Dirichlet boundary conditions the boundary value of the pressure p_r is assumed to be larger than zero. This assumption is essential insofar as the diffusion coefficient D_{\perp} remains finite on the boundary Γ , otherwise the derivatives of p would go to infinity on the boundary. Boundary conditions of the mixed type are obtained if the diffusion velocity on the boundary is smaller than or equal to the thermal velocity.

$$\mathbf{N} \cdot \left\{ \frac{p}{B^2 + \varepsilon g(x)p} \nabla_{\perp} p + \frac{1}{\varepsilon g} \nabla_{\parallel} p \right\} = \sigma p v_{th} \quad \text{on } \Gamma, \quad (13)$$

where \mathbf{N} is the normal vector to the boundary Γ .

In the case of $\Phi = \text{const}$ on Γ we find: $\mathbf{N} \cdot \mathbf{v}_c = 0$ on Γ . The convective motion does not carry plasma across the boundary. In the following we only con-

sider Dirichlet boundary conditions:

$$\begin{aligned} \Phi &= \Phi_r = 0 & \text{on } \Gamma, \\ p &= p_r = \text{const}, \quad p_r > 0 & \text{on } \Gamma. \end{aligned} \quad (14)$$

4. Existence Theorem

To abbreviate the notation we introduce two matrices A and D by

$$\begin{aligned} D_{ik} &= D_{\perp} \delta_{ik} + (D_{\parallel} - D_{\perp}) \frac{B_i B_k}{B^2}, \\ A_{ik} &= A_{\perp} \delta_{ik} + (A_{\parallel} - A_{\perp}) \frac{B_i B_k}{B^2} \end{aligned} \quad (15)$$

with

$$A_{\perp} = \frac{\varepsilon g(x)p}{B^2 + \varepsilon g(x)p}, \quad A_{\parallel} = 1$$

and

$$D_{\perp} = \frac{p}{B^2 + \varepsilon g(x)p}, \quad D_{\parallel} = \frac{1}{\varepsilon g(x)}.$$

The correlation between A_{ik} and D_{ik} is $D_{ik} = 1/\varepsilon g A_{ik}$. Furthermore we define a vector \mathbf{b} by

$$\begin{aligned} &\nabla \cdot \frac{\nabla p \times \mathbf{B}}{B^2 + \varepsilon g(x)p} \\ &= \nabla p \cdot \left(\mathbf{B} \times \nabla \frac{1}{B^2 + \varepsilon g(x)p} - \frac{1}{B^2 + \varepsilon g(x)p} \nabla \times \mathbf{B} \right) \\ &= \nabla p \cdot \mathbf{b}. \end{aligned} \quad (16)$$

The vector \mathbf{b} depends on εp and on its first derivatives $D^1 p$. It also depends continuously on the magnetic field and its first derivatives. Furthermore we assume $g(x)$ to be positive and continuously differentiable. With the aid of these definitions the system is modified to

$$\begin{cases} -\nabla \cdot A(p) \cdot \nabla \Phi = \mathbf{b} \cdot \nabla p, \\ -\nabla \cdot D(p) \cdot \nabla p - \mathbf{v}_c \cdot p \nabla - p \nabla \cdot \mathbf{v}_c = S \end{cases} \quad (17)$$

with $S > 0$, $n = p$ and boundary conditions (14).

The matrices A and D are positive definite:

$$\sum D_{ik} y_i y_k \geq d \sum y_i^2, \quad \sum A_{ik} y_i y_k \geq a \sum y_i^2$$

for all vectors $\mathbf{y} = \{y_i\}$

$$d = \min \{D_{\perp}, D_{\parallel}\}, \quad a = \min \{A_{\perp}, A_{\parallel}\}.$$

Since D_{\perp} and A_{\perp} are proportional to p , it is required that $p > 0$ in $G + \Gamma$, otherwise d, a would

become zero somewhere and the matrices would no longer be positive definite. On the other hand, only positive solutions of (17) are of any physical relevance. Therefore the basic question is: Under what condition does a unique solution of the system (17) exist with $p > 0$?

It can be easily shown that in case of source function $S = 0$ only the trivial solution $\Phi = 0$, $p = p_r$ exists. By multiplying the first equation of (17) with Φ and the second equation with $\ln p$ and integrating over the domain G we obtain

$$\begin{aligned} \iiint_G \nabla \Phi \cdot A \cdot \nabla \Phi \, d^3x &= \iiint_G \nabla p \cdot v_c \, d^3x, \\ \iiint_G \frac{1}{p} \nabla p \cdot D \cdot \nabla p \, d^3x &+ \iiint_G \nabla p \cdot v_c \, d^3x \\ &= \iiint_G S \ln p \, d^3x \end{aligned}$$

or

$$\begin{aligned} \iiint_G \frac{1}{p} \nabla p \cdot D \cdot \nabla p \, d^3x &+ \iiint_G \nabla \Phi \cdot A \cdot \nabla \Phi \, d^3x \\ &= \iiint_G S \ln p \, d^3x. \end{aligned} \quad (18)$$

In case of $S = 0$ the left side only vanishes if $p = p_r$ and $\Phi = 0$. Therefore it is expected that a nontrivial solution exists if the source term is sufficiently small.

The existence theorem is based on the Fixed-Point method of Schauder [3, 4]. For this purpose a mapping T is constructed which maps a class of positive functions n

$$N = \{n: n \geq p_r \text{ in } G, \quad n = p_r \text{ on } \Gamma\}$$

into another class of positive functions p

$$P = \{p: p \geq p_r \text{ in } G, \quad p = p_r \text{ on } \Gamma\}.$$

If P is a subset of N , a fixed points exists under certain conditions. In order to construct this mapping, the system (17) is linearized in the following way:

$$\begin{cases} -\nabla \cdot A(n) \cdot \nabla \Phi = \mathbf{b} \cdot \nabla n \\ -\nabla \cdot D(n) \cdot \nabla p - v_c \cdot \nabla p - p \nabla \cdot v_c = S \end{cases}. \quad (19)$$

Here A and D are analytic functions of n , and v_c is a linear functional of n . For any given $n \in N$ the system (19) is a linear system in p and Φ . Since the matrices A and D are positive definite, the two operators $M = \nabla \cdot A : \nabla$ and $L = \nabla \cdot D : \nabla$ are elliptic and the linear problem reduces to a standard

problem of second order differential equations. The solution of (19) yields the required mapping T :

$$T: n \rightarrow p.$$

The conditions of the Schauder Fixed-Point theorem are the following:

- (a) The set N has to be a closed convex subset of a Banach space,
- (b) T has to be a continuous mapping of N into itself, with the image TN being precompact.

This version of the Fixed-Point theorem is given in the book of Gilbarg-Trudinger. The main problem in the proof is condition (b). Let n be from the class $C_{1,x}$, the class of functions with Hölder-continuous first derivatives. The norm in this case is defined as the maximum norm,

$$\|n\|_{1,x} = \max |n - p_r| + \max |D^1 n| + H_x [D^1 n].$$

$D^1 n$ is the first-order derivative of n , and H_x is the Hölder coefficient. Any bounded subset of $C_{1,x}$ with $\|n\|_{1,x} < K_0 < \infty$ is a closed convex subset of a Banach space;

$$N = \{n \in C_{1,x} : n \geq p_r \quad \|n\|_{1,x} \leq K_0\}.$$

Having defined the class of functions n , the first equation of (19) reduces to a standard problem of elliptic differential equations. If the right-hand side of $M[\Phi] = \mathbf{b} \cdot \nabla n$ is Hölder continuous and the coefficients of M are bounded and from C_x , a unique solution $\Phi \in C_{2,x}$ exists [5]. In order to satisfy these conditions, the magnetic field is assumed to be bounded with bounded first derivatives;

$$B_{\min} < B < B_{\max}, \quad \left| \frac{\partial B_i}{\partial x_k} \right| < H B_{\max}.$$

H is a constant depending only on the domain G .

With respect to its arguments n , $D^1 n$, g , B_i , $\partial B_i / \partial x_i$ the inhomogeneous term $\mathbf{b} \cdot \nabla n$ is continuously differentiable, and therefore with B_i , n , $g \in C_{1,x}$ the inhomogeneous term $\mathbf{b} \cdot \nabla n$ belongs to C_x . The coefficients A_\perp , A_\parallel are also continuously differentiable with A_\perp , $A_\parallel \in C_x$ and $\partial A_\perp / \partial x_k \in C_x$. The coefficient A_\perp is bounded by

$$a = \min \left(1 - \frac{B^2(x)}{B^2(x) + \varepsilon g_{\min} p_r} \right) < A_\perp < 1,$$

which implies that $a = \min \{A_\perp, A_\parallel\}$ does not depend on K_0 . The bounds on A_\perp only depend on p_r

and not on the specific choice of $n(x)$. Also the upper bounds of the derivatives of A_\perp only depend on p_r and K_0 . The dependence on B_{\max} , g_{\max} , g_{\min} is of no importance for further consideration. The inverse operator M^{-1} can be expressed in terms of a Green's function $K(x, y)$:

$$\Phi = \iint_G K(x, x') \mathbf{b} \cdot \nabla n \, d^3 y. \quad (20)$$

The basic property of the solution Φ is the validity of the Schauder estimates [6, 7], which provide an upper bound for the norm $\|\Phi\|_{2,\alpha}$:

$$\|\Phi\|_{2,\alpha} < C(\|\Phi\|_0 + \|\mathbf{b} \cdot \nabla n\|_{0,\alpha}). \quad (21)$$

The constant C depends on the dimensions of the domain G and the bounds of the coefficients in (11). The maximum principle of elliptic equations [8] provides an upper bound of the norm $\|\Phi\|_0$:

$$\|\Phi\|_0 \leq \frac{C_1(p_r, K_0, \varepsilon)}{a} \|\mathbf{b} \cdot \nabla n\|_0. \quad (22)$$

Combining these two equations implies that the derivatives $D^1\Phi$ and $D^2\Phi$ are bounded by a constant which is proportional to the maximum derivatives of n , and therefore the following estimate holds:

$$|D^1\Phi| \leq K_2 \frac{K_0}{a}. \quad (23)$$

The constant K_2 depends on p_r , K_0 , ε , and on G . The essential properties of the coefficient K_2 are: $K_2 \rightarrow 0$ with $\varepsilon \rightarrow \infty$ and $K_2 \rightarrow \infty$ with $\varepsilon \rightarrow 0$. For every finite ε and K_0 given, there exists an upper bound on the first order derivative of Φ and therefore on the electric field $\nabla\Phi$. The existence of such an upper bound represents the essential difference to the ideal MHD model. The finite friction term ε prevents the electric field $\nabla\Phi$ from diverging and the plasma currents remain bounded.

In the second equation of (19) the convective velocity \mathbf{v}_c is a given quantity. This equation does not always have a unique solution. A unique solution only exists if a nontrivial solution of the homogeneous equation does not exist. In (19b) we define a new variable p by replacing p by $p + p_r$. The boundary condition for this new variable p is changed to $p = 0$ on Γ :

$$-L[p] - \mathbf{v}_c \cdot \nabla p - p \nabla \cdot \mathbf{v}_c = S + p_r \nabla \cdot \mathbf{v}_c. \quad (24)$$

Any solution of the homogeneous equation would satisfy the integral relation

$$-\iint_G \nabla p \cdot D \cdot \nabla p \, d^3 y + \iint_G \frac{p^2}{2} \nabla \cdot \mathbf{v}_c \, d^3 y = 0. \quad (25)$$

The following estimate holds:

$$\iint_G \nabla p \cdot D \cdot \nabla p \, d^3 y \geq d \iint_G |\nabla p|^2 \, d^3 y \geq d \lambda_0 \iint_G p^2 \, d^3 y, \quad (26)$$

where λ_0 is the lowest eigenvalue of the Laplace operator in G . If $\nabla \cdot \mathbf{v}_c$ can be made small enough, such that

$$\max |\nabla \cdot \mathbf{v}_c| < 2d \lambda_0, \quad (27)$$

(25) cannot be satisfied by $p \neq 0$ and the homogeneous equation has no nontrivial solution. With $\nabla \cdot \mathbf{v}_c = \mathbf{b} \cdot \nabla \Phi$ we obtain

$$\max |\nabla \cdot \mathbf{v}_c| \leq \max |\mathbf{b}| K_2 \frac{K_0}{a}. \quad (28)$$

Therefore, by combining (27) and (28) a condition for a unique solution is

$$\max |\mathbf{b}| K_2 K_0 < ad \lambda_0. \quad (29)$$

The lowest eigenvalue of the Laplacian λ_0 only depends on the domain G . For large ε the minimum $d = 1/\varepsilon g_{\max} \min A_\perp$ is of the order $1/\varepsilon$, whereas the left-hand side of (29) scales as $1/\varepsilon^2$. If the friction term ε is large enough, the condition for a unique solution can be satisfied.

In order to obtain positive solutions of (24), the effective source term $S + p_r \nabla \cdot \mathbf{v}_c$ has to be positive. This certainly the case if

$$p_r \max |\nabla \cdot \mathbf{v}_c| \leq S \quad \text{everywhere in } G. \quad (30)$$

This condition is certainly satisfied if

$$p_r \max |\mathbf{b}| K_2 \frac{K_0}{a} < S_{\min}. \quad (31)$$

To prove the existence of positive solutions p , use is made again of the maximum principle of elliptic operators. With (24) written in the form

$$\begin{aligned} -\{L[p] + \mathbf{v}_c \cdot \nabla p + p(\nabla \cdot \mathbf{v}_c - \mu)\} &= S_1 + \mu p, \\ \mu &= \max |\nabla \cdot \mathbf{v}_c|, \quad S_1 = S + p_r \nabla \cdot \mathbf{v}_c, \end{aligned} \quad (32)$$

the solution is found by the following iteration process:

$$-L^*[p_{n+1}] = S_1 + \mu p_n. \quad (33)$$

The iteration process converges if inequality (29) is satisfied. L^* is the operator in braces. For every $p_n \geq 0$ the right hand side of (32) is positive because of

$$\nabla \cdot v_c - \mu \leq 0, \quad (34)$$

the maximum principle holds for the operator L^* and the solution p_{n+1} is nonnegative, and consequently the solution of (24) is nonnegative. According to the existence theorem of the Dirichlet problem the solution p belongs to the space $C_{2,\alpha}$. Following the Schauder estimates the norm $\|p\|_{2,\alpha}$ is bounded by the norm $\|p\|_0$ and the maximum of the inhomogeneous term:

$$\|p\|_{2,\alpha} \leq C_2(\|p\|_0 + \|S_1\|_{0,\alpha}). \quad (35)$$

The constant C_2 depends on ε , p_Γ , K_0 and the domain G .

Here we need further estimates on $\|p\|_0$. From the equation $L^*[p] = S_1 + \mu p$ the maximum principle yields

$$\max |p| \leq C_3(\varepsilon, p_\Gamma, K_0, G) \frac{\max |S_1 + \mu p|}{d} \quad (36)$$

or

$$\|p\|_0 = \max |p| \leq \frac{1}{1 - C_3\mu/d} \frac{C_3}{d} \max |S_1|. \quad (37)$$

The denominator remains finite if μ is small enough: $1 - C_3\mu/d > 0$. The result for the $\|p\|_{2,\alpha}$ norm is

$$\begin{aligned} \|p\|_{2,\alpha} &\leq C_2 \left\{ \frac{C_3}{d - C_3\mu} \|S_1\|_{0,\alpha} + \|S_1\|_{0,\alpha} \right\}, \\ \|p\|_{2,\alpha} &\leq C_4(\varepsilon, p_\Gamma, K_0, G) \|S_1\|_{0,\alpha}. \end{aligned} \quad (38)$$

The final result is: The mapping $T: n \rightarrow p$ maps the bounded set of functions

$$N = \{n \in C_{1,\alpha}, \quad \|n\|_{1,\alpha} \leq K_0\}$$

and

$$P = \{p \in C_{2,\alpha}, \quad \|p\|_{2,\alpha} \leq C_4 \|S_1\|_{0,\alpha}\}.$$

By a suitable choice of the source term S_1 the norm $\|p\|_{2,\alpha}$ can be made so small that P is a subset of N which implies

$$C_4 \|S_1\|_{0,\alpha} \leq K_0. \quad (39)$$

In summary, three conditions have to be satisfied:

Condition (29) for a unique solution of (24),

Condition (30) or (31) for a positive solution of (24);

Condition (39) in order to make P a subset of N .

These conditions determine the parameter range of S , ε , and K_0 , where a solution of the nonlinear problem exists. The accessible parameter regime of K_0 is bounded by condition (31); with $S \rightarrow 0$ this yields the trivial solution $p = p_\Gamma$, $\Phi = 0$. For a given K_0 , condition (31) can be satisfied if we choose ε large enough. Since K_2 scales as $1/\varepsilon^2$, this is always possible. T maps bounded sets in $C_{1,\alpha}$ into bounded sets in $C_{2,\alpha}$ which are precompact in $C_{1,\alpha}$ and C_2 according to Arzela's theorem. As shown in Gilbarg-Trudinger [9] this property is essential to prove the continuity of the mapping T . The proof follows the same line as given in their book and the final statement can be made:

Let G be a bounded domain with a smooth boundary Γ . The boundary conditions are:

$$\Phi = 0 \text{ on } \Gamma, \quad p = p_\Gamma > 0 \text{ on } \Gamma.$$

The magnetic field may be continuous with continuous first derivatives. Under these assumptions the system (12) has a unique solution $\Phi \in C_{2,\alpha}$ and $p \in C_{2,\alpha}$ if by a suitable choice of ε the conditions (29) and (31) are met and the source term S_1 is small enough so that $\|p\|_{2,\alpha} < K_0$ is satisfied.

In more physical terms condition (29) requires the convective velocity to be small. A small convective velocity results from large friction forces. In the limit $\varepsilon \rightarrow \infty$ or $g p \varepsilon > B^2$ the system (12) is simplified to

$$\begin{cases} -\Delta \Phi = \frac{1}{\varepsilon} \nabla \cdot \frac{\nabla p \times \mathbf{B}}{g p} \\ -\nabla \cdot \left\{ \frac{1}{g} \nabla p \right\} - \varepsilon \nabla \cdot p v_c = \varepsilon S \end{cases} \quad (40)$$

In this limit the convective velocity is of the order $1/\varepsilon^2$. If we choose ε large enough, the conditions (29) and (30) can always be satisfied. With decreasing friction the convective velocity grows and at a critical value of ε the conditions (29) for uniqueness of solutions are no longer met. At this point a bifurcation may occur and several solutions could exist beyond this limit. Violation of condition (30) means that the effective source term S_1 may become negative. If a negative pressure arises, however, in a subdomain of G this destroys the ellipticity of the

differential operators. The other limit $\varepsilon \rightarrow 0$ leads to $\nabla_{\parallel} p \rightarrow 0$. In this limit the ellipticity of the system is also destroyed and the plasma and magnetic field are closely coupled again.

5. Low-beta Expansion

In system (12) the magnetic field is a given quantity. As can be seen from (15), the derivatives of \mathbf{B} enter into the problem. The plasma current \mathbf{j} depends on $\nabla\Phi$ and ∇p . In dimensionless units it can be written

$$\mathbf{j} = -\frac{\varepsilon g p}{B^2 + \varepsilon g(x)p} \nabla_{\perp} \Phi - \nabla_{\parallel} \Phi - \frac{1}{B^2 + \varepsilon g(x)p} \nabla p \times \mathbf{B}. \quad (41)$$

From this current another magnetic field \mathbf{B}_1 can be calculated. The classical result of this problem states that \mathbf{B} is bounded and has bounded derivatives if the current \mathbf{j} is Hölder-continuous. But this is the case since Φ and p are the solution of a uniformly elliptic system. Our solution procedure again provides a mapping of one set of magnetic fields \mathbf{B} into another set \mathbf{B} . In order to prove the existence of a fixed point, the procedure follows the same lines as described above for Φ and p . The magnetic field \mathbf{B} is split into a vacuum field \mathbf{B}_v and a field \mathbf{B} generated by the plasma currents. A Banach space M_B of functions \mathbf{B} is constructed with the norm

$$\|\mathbf{B}\|_{1,x} = \max |\mathbf{B}| + \max |D\mathbf{B}| + H_x[D\mathbf{B}].$$

The derivatives of \mathbf{B} have to be Hölder-continuous. We consider a bounded subset $M \subset M_B$ which is small enough to satisfy conditions (24) and (25). This subset is compact and convex. The mapping $\mathbf{B} \rightarrow \mathbf{B}_1$ defined by $\nabla \times \mathbf{B}_1 = \mathbf{j}[\mathbf{B}]$ maps M into itself if j can be made small enough. This can be achieved by a sufficiently small source function S . As has been shown in the former section the upper bounds of $|\nabla\Phi|$ and $|\nabla p|$ depend on S and can be made sufficiently small by a proper choice of the source function. Small S implies a small plasma pressure. We may therefore conclude:

Within the dissipative MHD-model there exists a selfconsistent equilibrium if the plasma pressure is sufficiently small.

To illustrate the situation let us return to the limit of large friction. If ε is made large enough, $\nabla \cdot p \mathbf{v}_c$ can be neglected in the second equation. The pres-

sure only depends on S and the magnetic field does not enter into the equation for p . The problem reduces to

$$\nabla \times \mathbf{B} = \mathbf{j}; \quad \nabla \cdot \mathbf{B} = 0, \quad (42)$$

$$\mathbf{j} = -\nabla\Phi - \frac{1}{g\varepsilon} \nabla \ln p \times \mathbf{B}, \quad (43)$$

$$-\Delta\Phi = \frac{1}{\varepsilon} \nabla \cdot \frac{1}{g} (\nabla \ln p \times \mathbf{B}). \quad (44)$$

Since \mathbf{j} depends linearly on $\nabla \ln p$, it can be made arbitrarily small for any given magnetic field \mathbf{B} . Consequently, the iterative solution of (42) will converge if $|\nabla \ln p|$ is made small enough.

6. Closed Magnetic Field Lines

The critical problem of the ideal MHD equilibrium is the behaviour at rational magnetic surfaces. The current density diverges beyond limits if the $\oint d\ell/B = \text{const}$ condition is not satisfied. The dissipative model eliminates this divergence since $\nabla\Phi$ and ∇p are bounded. In the case of small friction, however, the current density may become very large and may lead to rapid destruction of the magnetic surfaces.

To illustrate the behaviour at rational magnetic surfaces we consider the extreme case where all magnetic field lines are closed. The rotational transform is a constant and may be a rational number ($\iota = n/m$). Furthermore we go to the limit of small ε ($\varepsilon g p \ll B^2$). The diffusion parallel to the magnetic field becomes very large compared with the perpendicular diffusion. Consequently pressure and potential vary only little along the magnetic field lines and can be written in the form

$$p = p_0 + p_1, \quad \Phi = \Phi_0 + \Phi_1.$$

p_0 and Φ_0 are independent of the parallel coordinate l but may still depend on ε .

$$\nabla_{\parallel} \Phi_0 = 0, \quad \nabla_{\parallel} p_0 = 0.$$

Equations (12) are reduced to

$$\begin{aligned} -\nabla \cdot \left\{ \frac{\varepsilon g p_0}{B^2} \nabla_{\perp} \Phi_0 + \nabla_{\parallel} \Phi_1 \right\} &= \nabla \cdot \frac{\nabla p_0 \times \mathbf{B}}{B^2}, \\ -\nabla \cdot \left\{ \frac{p_0}{B^2} \nabla_{\perp} p_0 + \frac{1}{g\varepsilon} \nabla_{\parallel} p_1 \right\} - \nabla \cdot p_0 \mathbf{v}_c &= S, \\ \mathbf{v}_c &= \frac{\nabla \Phi_0 \times \mathbf{B}}{B^2}. \end{aligned} \quad (45)$$

The terms containing p_1 and Φ_1 are eliminated by averaging along the magnetic field lines. The averaging process is defined by

$$\bar{f} = \frac{\oint f dl/B}{\oint dl/B}.$$

With

$$\nabla \cdot \nabla_{\parallel} \Phi_1 = \nabla \cdot \frac{\mathbf{B}}{B^2} (\mathbf{B} \cdot \nabla \Phi_1) = \mathbf{B} \cdot \nabla \left(\frac{\mathbf{B}}{B^2} \cdot \nabla \Phi_1 \right)$$

we obtain

$$\nabla \cdot \nabla_{\parallel} \Phi_1 = 0.$$

As perpendicular coordinates we introduce the two Clebsch variables ψ and χ defined by $\mathbf{B} = \nabla\psi \times \nabla\chi$. The three-dimensional volume element in this coordinate system is

$$d^3x = \frac{dl}{B} d\psi d\chi.$$

The averaging process defines two new second order differential operators M_{\perp} and L_{\perp} by

$$M_{\perp}[\Phi_0(\psi, \chi)] = \nabla \cdot \frac{g p_0}{B^2} \nabla_{\perp} \Phi_0(\psi, \chi), \quad (46)$$

$$L_{\perp}[p_0(\psi, \chi)] = \nabla \cdot \frac{p_0}{B^2} \nabla_{\perp} p_0(\psi, \chi). \quad (47)$$

The operators $-M_{\perp}$ and $-L_{\perp}$ are hermitian and positive definite as can be seen from

$$\begin{aligned} -(\Phi_0, M_{\perp}[\Phi_0]) &= - \iint \Phi_0 M_{\perp}[\Phi_0] d\psi d\chi \\ &= - \iiint_G \Phi_0 \nabla \cdot \frac{g p_0}{B^2} \nabla_{\perp} \Phi_0 d^3x \\ &= \iiint_G \nabla_{\perp} \Phi_0 \cdot \frac{g p_0}{B^2} \nabla_{\perp} \Phi_0 d^3x \\ &> 0. \end{aligned} \quad (48)$$

A similar relation holds for the operator L_{\perp} . In averaging the other terms along the magnetic field line we assume the magnetic field to be a vacuum field. This could be considered as the first step of a low- β expansion. As a parallel coordinate the magnetic potential $U = B dl$ is introduced. It then follows that

Here $q = \oint dl/B$. The brackets $[,]$ are the well known Poisson brackets. In a similar procedure the average on $\nabla \cdot p_0 \mathbf{v}_c$ yields

$$\oint \frac{dl}{B} \nabla \cdot p_0 \frac{\nabla \Phi_0 \times \mathbf{B}}{B^2} = p_0[\Phi_0, q] + q[\Phi_0, p_0].$$

The averaged equations are

$$\begin{cases} -\varepsilon M_{\perp}[\Phi_0] = [p_0, q] \\ -L_{\perp}[p_0] - p_0[\Phi_0, q] - q[\Phi_0, p_0] = \bar{S} \end{cases} \quad (50)$$

The mathematical structure of these equations is of the same type as the full system (12). Therefore the same conclusions with respect to existence and uniqueness of solutions can be drawn.

7. Relation to Thermal Convection

The system (50) has a close similarity to the 2-dimensional Bénard problem of a fluid heated from below. Let $U(x, y)$ be a gravitational potential and $V\mathbf{r}$ a viscosity term. The equations of thermal convection are

$$\nabla p = -V\mathbf{r} + \varrho(T) \nabla U, \quad (51)$$

$$-\nabla \cdot \alpha \nabla T + \nabla \cdot T\mathbf{v} = Q, \quad \nabla \cdot \mathbf{v} = 0. \quad (52)$$

All quantities are assumed to be independent of the z -coordinate. Inertial terms are also neglected in this approximation $\varrho(T) = \varrho_0 + \alpha T$, ($\alpha < 0$), is the temperature-dependent density of the fluid. Introducing the stream function Φ by

$$\mathbf{v} = \nabla \Phi \times \mathbf{e}$$

with \mathbf{e} being the unit vector in z -direction, one can modify the first equation to

$$-\nabla \cdot (\mathbf{e} \times V(\mathbf{v})) = \alpha \mathbf{e} \cdot (\nabla T \times \nabla U) \quad (53)$$

or

$$-\nabla \cdot (\mathbf{e} \times V(\mathbf{v})) = \alpha [T, U].$$

The derivatives in the Poisson brackets are taken in x, y coordinates. The temperature equation can also be rewritten in terms of Poisson brackets:

$$-\nabla \cdot \alpha \nabla T - [T, \Phi] = Q. \quad (54)$$

$$\oint \frac{dl}{B} \nabla \cdot \frac{\nabla p_0 \times \mathbf{B}}{B^2} = \int dU \left\{ \frac{\partial p_0}{\partial \psi} \frac{\partial}{\partial \chi} \left(\frac{1}{B^2} \right) - \frac{\partial p}{\partial \chi} \frac{\partial}{\partial \psi} \left(\frac{1}{B^2} \right) \right\} = \frac{\partial p_0}{\partial \psi} \frac{\partial q}{\partial \chi} - \frac{\partial p_0}{\partial \chi} \frac{\partial q}{\partial \psi} = [p_0, q]. \quad (49)$$

The mathematical structure of these equations is the same as in (50), except that the operator

$$-\nabla \cdot (\mathbf{e} \times V(\nabla\Phi \times \mathbf{e}))$$

is of 4th order in x, y , whereas $M_{\perp}(\Phi_0)$ is of second order. The role of the gravitational potential is played by the function $q = \oint dI/B$. In the limit $\varepsilon = 0$, q -surfaces and pressure surfaces coincide. Owing to this similarity to thermal convection, similar phenomena as known from this problem are expected in a plasma:

unique solution at large friction (corresponding to a low Rayleigh number); bifurcation and the onset of convection if the friction term decreases.

In a plasma with shear this onset of convection is localized to the neighbourhood of rational magnetic surfaces. The localisation depends on the magnitude of shear and friction terms.

8. Discussions and Conclusions

Deriving the two equations for p and Φ , we assumed an isothermal plasma with $T = \text{const}$. This restriction is not very essential, it can easily be abandoned by including the equation of thermal transport

$$\nabla \cdot \mathbf{q} + \nabla \cdot 3/2(nT\mathbf{v}) + nT\nabla \cdot \mathbf{v} = Q, \quad (55)$$

\mathbf{q} is the heat flux:

$$\mathbf{q} = -\kappa_{\parallel} \nabla_{\parallel} T - \kappa_{\perp} \nabla_{\perp} T,$$

κ_{\parallel} , κ_{\perp} being the thermal conductivity, and Q is the heat source. This would add another elliptic equation for T to the system (11) and the procedure for proving the existence of a solution would remain essentially the same.

The low- β expansion as proposed by Spitzer does not exclude the formation of magnetic islands and

the ergodisation of field lines. Recently, this effect was extensively discussed by Boozer and Reiman [10]. In the iteration scheme discussed above similar effects may occur. The solvability of the system (11) or (12), however, does not depend on topological constraints on the magnetic field. Island formation or the destruction of magnetic surfaces would enhance the effective transport perpendicular to the magnetic field, as can be seen from (11). With the source function S fixed, this would lead to a decrease of $|\nabla\Phi|$ and $|\nabla p|$ and therefore to a reduction of the plasma currents \mathbf{j} via (9). Since the plasma currents are responsible for the destruction of the magnetic surfaces, this introduces a stabilizing feedback mechanism into the iteration scheme. It is expected that destruction of magnetic surfaces leads to faster convergence of the iteration procedure than without destruction, but it results in a lower value of β .

In practical cases of a fusion plasma the friction term $\alpha\mathbf{v}$ is a small effect. We have to expect either several solutions of the system (12) or non-stationary convection as is well-known from the Bénard problem. But here other effects such as inertial forces and viscous forces neglected so far may play an important role, which is beyond the scope of this paper. The present model might be useful for analyzing the decoupling of the plasma and magnetic field and the convective processes arising around rational magnetic surfaces. The effect of field line ergodisation can be studied in this model. In a further study viscous effects will be included and the stability analysis of the convective state will be undertaken.

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